THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050B Mathematical Analysis I (Fall 2016)
Suggested Solutions to Homework 6

1. Let $A \subseteq \mathbb{R}$ be nonempty, $f : A \to \mathbb{R}$, $A^+ := A \cap (x_0, \infty)$, $A^- := A \cap (-\infty, x_0)$, and $x_0 \in \mathbb{R}$ be a cluster point of both A^+ and A^-.

   Show that $\lim_{x\to x_0} f(x) = \infty$ if and only if $\lim_{x\to x_0^+} f(x) = \infty$ and $\lim_{x\to x_0^-} f(x) = \infty$, and the corresponding result for $-\infty$.

   Proof. " $\Longrightarrow$ " Assume $\lim_{x\to x_0} f(x) = \infty$. Let $M > 0$ be given. Since $\lim_{x\to x_0^+} f(x) = \infty$, there exists $\delta_1 > 0$ such that for all $x_0 < x < x_0 + \delta_1$, $x \in A$, $f(x) > M$.

   Similarly, since $\lim_{x\to x_0^-} f(x) = \infty$, there exists $\delta_2 > 0$ such that for all $x_0 - \delta_2 < x < x_0$, $x \in A$, $f(x) > M$.

   Now take $\delta := \min\{\delta_1, \delta_2\} > 0$, then for $0 < |x - x_0| < \delta$, $x \in A$, $f(x) > M$. Therefore $\lim_{x\to x_0} f(x) = \infty$, since $M > 0$ is arbitrary.

   " $\Longleftarrow$ " Assume $\lim_{x\to x_0^+} f(x) = \infty$ and $\lim_{x\to x_0^-} f(x) = \infty$. Let $M > 0$ be given. Since $\lim_{x\to x_0} f(x) = \infty$, there exists $\delta > 0$ such that for $0 < |x - x_0| < \delta$, $x \in A$, we have $f(x) > M$.

   Now for the same $\delta > 0$, it is true that $f(x) > M$ for $x_0 < x < x_0 + \delta$ $x \in A$ and that $f(x) > M$ for $x_0 - \delta < x < x_0$, $x \in A$. Hence $\lim_{x\to x_0^+} f(x) = \infty$ and $\lim_{x\to x_0^-} f(x) = \infty$.

   The case $-\infty$ is similar.

   □

4(c). Compute

$$\lim_{x\to\infty} \frac{\sqrt{x} - 5}{\sqrt{x} + 3}$$

Solution:

By MATH 1010 we claim that the limit is 1. Let $\epsilon > 0$ be given. Take $t := \frac{64}{\epsilon^2} > 0$. Then for any $x > t$, we have:

$$\begin{aligned}\left|\frac{\sqrt{x} - 5}{\sqrt{x} + 3} - 1\right| &= \frac{8}{\sqrt{x} + 3} \\ &< \frac{8}{\sqrt{t} + 3} \\ &< \frac{8}{\sqrt{t}} \\ &= \epsilon\end{aligned}$$

Hence

$$\lim_{x\to\infty} \frac{\sqrt{x} - 5}{\sqrt{x} + 3} = 1$$

4(d). Compute

$$\lim_{x\to\infty} \frac{\sqrt{x}-x}{\sqrt{x}+x}$$

Solution:

By MATH 1010 we claim that the limit is -1. Let $\epsilon > 0$ be given. Take $t := \frac{4}{\epsilon^2} > 0$. Then for any $x > t$, we have:

$$\begin{aligned}
\left|\frac{\sqrt{x}-x}{\sqrt{x}+x}+1\right| &= \frac{2\sqrt{x}}{\sqrt{x}+x} \\
&= \frac{2}{1+\sqrt{x}} \\
&< \frac{2}{1+\sqrt{t}} \\
&< \frac{2}{\sqrt{t}} \\
&= \epsilon
\end{aligned}$$

Hence

$$\lim_{x\to\infty} \frac{\sqrt{x}-x}{\sqrt{x}+x} = -1$$